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Double-diffusive convection with large variable gradients

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The onset of double-diffusive convection is discussed for a layer of fluid in which the vertical salinity gradient varies with depth and for which the thermal and saline Rayleigh numbers R and R_S are large. These conditions are similar to those that exist in a solar pond prior to the onset of any instability. It is shown that when convection occurs it takes the form of an overstable mode and is essentially confined to a narrow region of vertical extent $\sim R_S^{-\frac{1}{14}} \times$ depth of the fluid layer, centred at the critical depth where the salt gradient is smallest. The leading terms in asymptotic expansions of the ratio R/R_S, the frequency of oscillation p and the horizontal wavenumber a are determined for $R_S \gg 1$. The results predicted by the theory are shown to be in good agreement with numerical results and with observations of solar ponds.

1. Introduction

A layer of fluid that is stratified by temperature and by a solute (salt or sugar, for example) may become unstable even though the net density gradient increases with depth so that the layer is statically stable. This phenomenon is known as double-diffusive convection. It is now well documented (see Huppert & Turner 1981), and the stability characteristics of the layer have been explored for a variety of boundary conditions and parameter regimes. Most theoretical research in this field makes the idealization that the vertical temperature and solute concentration gradients, prior to the onset of instability, are constant. This is rarely the case in either laboratory (Shirtcliffe 1967, 1969) or natural systems (Turner 1973; Zangrando 1979). In an attempt to provide a more realistic theoretical model we shall consider the effect of a variable vertical solute concentration gradient on the stability of the layer.

The form of the instability in a double-diffusive system depends upon whether the driving energy comes from the substance with higher or lower diffusivity. In the case in which, for example, hot salty water lies over cold, fresher water, the instability takes the form of 'fingers' and is monotone in time. On the other hand, when cold fresh water lies above hot salty water an oscillatory instability occurs, which gives rise to distinct homogeneous layers separated by relatively sharp density interfaces. We shall be concerned only with this second kind of instability in the present paper.

It has been known since the turn of the century that certain lakes are hotter and saltier at the bottom than at the top (Hauser 1976). Typically, the vertical salt and temperature concentrations vary smoothly over much of the depth of the lake, but there are relatively narrow layers that exhibit the step-like profiles associated with double-diffusive instability. In these lakes the salinity gradient is maintained by salt leaching at the bottom, perhaps from a salt bed, and by the infusion of fresher river or sea water near the surface. Lake Vanda in Antarctica is a particularly well-documented example (Wilson & Wellman 1962; Hoare 1966). These lakes are heated by solar radiation, which penetrates to the depths where the reflected energy is absorbed. In the undisturbed (non-convecting) regions well away from the surface and the bottom the temperature gradient may be regarded as linear, but it is not a good approximation to regard the salinity gradient as linear.

It has been reported that the temperature at the bottom of such lakes may exceed the surface temperature by as much as 40 or 50 °C. This has led to the construction of 'solar ponds' as a potential energy source (Weinberger 1964; Zangrando 1979; Tabor 1980). They appear to be economically viable, at least as a direct heating source, provided of course that sufficient radiation is received from the sun and provided that convective overturning does not occur. If overturning occurs the temperature gradient is soon destroyed and the region near the bottom ceases to be a heat reservoir. Temperatures in excess of 90 °C have been reported at the bottom of stable solar ponds. Clearly a good criterion for the stability of solar ponds is of paramount importance and this is one of the motivations of the present paper. Observations of a working solar pond reported by Zangrando (1979) show that the vertical salinity gradient varies considerably with depth. A typical set of observations is reproduced as figure 1.

We consider a layer of fluid in which the salt concentration and temperature increase with depth in such a way that the temperature gradient is uniform but the salinity gradient varies with depth. The analysis that follows may be applied also to a base state that is quasi-static, provided that its typical timescale is much longer than that of the disturbance. We shall not concern ourselves with how this steady state is set up and maintained but merely seek to define its stability characteristics. For constant temperature and salinity gradients Baines & Gill (1969) showed that, provided the horizontal surfaces of the layer are stress-free and perfect conductors of both heat and salt, the layer becomes unstable to small perturbations when

$$R = \frac{\sigma+\tau}{\sigma+1} R_s + \frac{27\pi^4}{4} \frac{(1+\tau)(\sigma+\tau)}{\sigma}, \tag{1}$$

where R and R_s are the thermal and saline Rayleigh numbers, σ is the Prandtl number and τ is the ratio of saline diffusivity to thermal diffusivity. When instability occurs it does so as an overstability (i.e. it oscillates in time) whose vertical scale is that of the fluid layer. This contradicts observations of natural and man-made solar ponds in which the overturning is confined to relatively narrow depth bands. Indeed, all observations of convection in this regime indicate that the scale of the step-like profiles of the instability is very much smaller than that of the whole fluid layer. The constant-gradient model is therefore severely deficient in its prediction of the scale of the motion. Nevertheless, the assumption of linear gradients is the basis for a criterion for the stability of solar ponds proposed by Weinberger (1964) before Baines & Gill's work. Hauser (1976) later demonstrated that when Weinberger's criterion is written in terms of R, R_s it becomes

$$R \leqslant \frac{\sigma+\tau}{\sigma+1} R_s. \tag{2}$$

We shall show in §3 that Weinberger's criterion (2) corresponds to Baines & Gill's criterion (1), if the depth of the pond is assumed to be infinite. For many natural systems Hauser (1976) has shown that $R_s \sim 10^{10}$–10^{15} in the non-convecting regions, and Zangrando (1979) has shown that for man-made solar ponds R_s is typically $O(10^{10})$. In the limit $R_s \to \infty$ criteria (1) and (2) are identical.

Theoretical work on a layer with variable gradients is confined to some numerical computations at present being undertaken at Sandia Laboratories in New Mexico by Zangrando and Bertram (private communication). Their scheme assumes constant temperature gradient, variable salinity gradient, and stress-free horizontal surfaces. They have obtained satisfactory results at moderate values of R_s by expanding the dependent variables in Fourier series in the vertical variable and truncating at a suitably high number of terms. As R_s increases, the scale of the disturbance decreases, and it becomes increasingly difficult to resolve. Bertram reported that 75 terms are needed to obtain 4-figure accuracy for $R_s = 10^{12}$.

In order to complement Zangrando and Bertram's results and in order to gain some insight into the stability of solar ponds we assume in this paper that $R_s \gg 1$ and look for an asymptotic solution of the governing equations. These equations are given in §2. Baines & Gill's (1969) results for the constant-gradient problem, together with the results for a layer of infinite depth, are given in §3. The main analysis is given in §4. We are able to obtain the scale and structure of the disturbance and the critical depth at which it is concentrated. The first approximation to the stability criterion is shown to be identical with Weinberger's criterion (2), and a second approximation is given. The theoretical results are compared briefly in §5, with Zangrando and Bertram's numerical results and with observations of natural and man-made solar ponds.

2. The governing equations

We consider a layer of fluid of depth d in which the values of the temperature T' and salinity S' at the lower boundary exceed their values at the upper boundary by $\Delta T'$, $\Delta S'$ respectively. The depth d will be used as a typical lengthscale, while $\Delta T'$, $\Delta S'$ characterize the temperature and salinity variations. We use dimensionless Cartesian coordinates (x, z) with x measured horizontally and z measured vertically downwards from the upper surface $z = 0$.

We shall assume that the boundary conditions are such that a steady state may be set up in which the fluid is motionless and in which the vertical temperature gradient is constant (by definition it is therefore unity) and the salinity gradient is a function $G(z)$ of depth. Thus, in the base state

$$\frac{dT}{dz} = 1, \quad \frac{dS}{dz} = G(z), \tag{3}$$

where non-primed variables are dimensionless.

We define t to be a dimensionless time scaled on $d^2\kappa_T^{-1}$, where κ_T is the coefficient of thermal diffusivity, and adopt $\kappa_T d^{-1}$ as a typical velocity scale. Then the linearized Boussinesq perturbation equations are (cf. Baines & Gill 1969)

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi = -R\frac{\partial T}{\partial x} + R_s\frac{\partial S}{\partial x}, \tag{4a}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T = -\frac{\partial \psi}{\partial x}, \tag{4b}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S = -G(z)\frac{\partial \psi}{\partial x}, \tag{4c}$$

where ψ is a stream function with velocity given by

$$\mathbf{u} = (u, w) = \left(-\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x}\right) \tag{4d}$$

and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial z^2$. The Prandtl number σ is defined by $\sigma = \nu/\kappa_T$, where ν is the coefficient of kinematic viscosity, and $\tau = \kappa_S/\kappa_T$, where κ_S is the coefficient of saline diffusivity. For simplicity we have ignored the variation of these coefficients with temperature. A typical value of τ for salty water is 0·01, while σ varies between about 3 at 70 °C to about 7 at 20 °C in a 10 % salt concentration. The thermal and saline Rayleigh numbers are defined by

$$R = \frac{\alpha g \Delta T d^3}{\nu \kappa_T}, \quad R_S = \frac{\beta g \Delta S d^3}{\nu \kappa_T} \tag{5a, b}$$

respectively, where g is the acceleration due to gravity and α, β are the coefficients of cubical expansion with respect to thermal and saline variations.

When S, T are eliminated from (4) we obtain the following equation for ψ only:

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi = \sigma R_S \frac{\partial^2}{\partial x^2}\left\{\lambda\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\psi - \left(\frac{\partial}{\partial t} - \nabla^2\right)(G\psi)\right\}, \tag{6}$$

where $\lambda = R/R_S$. We shall suppose that R_S is fixed and seek to determine the maximum value of λ for which no unstable solutions to (6) exist, subject to certain boundary conditions to be specified later.

3. The constant-gradient solution

It is useful for the subsequent development of the theory for variable salinity gradient G to recall the main results for constant gradients in which $G = 1$. The simplest case has stress-free boundaries at $z = 0, 1$ (see Baines & Gill 1969), but other boundary conditions have also been considered (Nield 1967). For stress-free boundaries a solution of (6) is sought in the form

$$\psi \sim \exp\{i(pt + ax)\}\sin(n\pi z), \tag{7}$$

where n is an integer and a is the horizontal wavenumber. For neutrally stable solutions $\mathrm{Im}\{p\} = 0$, but it turns out that the principle of the exchange of stabilities does not hold in general for the most unstable mode, and $\mathrm{Re}\{p\} \neq 0$. When (7) is substituted into (6) we find that

$$ip^3 + ((\sigma+\tau+1)k^2p^2 - ip((\sigma+\sigma\tau+\tau)k^4 - (\lambda-1)\sigma R_S a^2 k^{-2}) - (\sigma\tau k^6 + (1-\tau\lambda)R_S\sigma a^2) = 0, \tag{8}$$

where $k^2 = a^2 + n^2\pi^2$. For $R_S > 0$ the mode that first becomes unstable as λ increases is $n = 1$, and it does so when

$$\lambda = \frac{\sigma+\tau}{\sigma+1} + \frac{(\sigma+\tau)(1+\tau)}{\sigma R_S}\frac{(\pi^2+a^2)^3}{a^2}, \tag{9a}$$

with
$$p^2 = (\sigma\tau+\sigma+\tau)(a^2+\pi^2) - (\lambda-1)\sigma R_S a^2(a^2+\pi^2)^{-1}. \tag{9b}$$

The value of the horizontal wavenumber a that minimizes λ is easily found to be $a = a_c = \sqrt{\tfrac{1}{2}}\,\pi$, and the critical values λ_c, P_c of λ, p are given by

$$\lambda_c = \frac{\sigma+\tau}{\sigma+1} + \frac{(\sigma+\tau)(1+\tau)}{\sigma R_S}\frac{27\pi^4}{4}, \tag{10a}$$

$$p_c^2 = \tfrac{3}{2}\pi^2(\sigma\tau+\sigma+\tau) - \tfrac{2}{3}\sigma R_S(\lambda_c - 1). \tag{10b}$$

For $R_s \gg 1$ with $\lambda \sim 1$ we have

$$\lambda_c \sim (\sigma+\tau)(\sigma+1)^{-1}, \quad p_c^2 \sim \tfrac{2}{3}\sigma R_s(1-\lambda_c) \tag{11a, b}$$

to leading order in R_s. (Note that, since $\tau < 1$, $\lambda_c < 1$, $p_c^2 > 0$.)

We shall show, in §4, that if G varies with depth and $R_s \gg 1$, then the disturbance is confined to the neighbourhood of a certain critical depth. Provided that the critical depth is not close to one of the boundaries, $z = 0, 1$, the boundaries appear to be at infinity on the scale of the disturbance, and G is constant to a first approximation. One might expect that the analysis for constant gradients in an unbounded fluid is relevant, and, in anticipation of that, we now outline the results for that case (Walin (1964) has also examined this case, but he did not give the results needed here.) We look for a solution of (6) in the form

$$\psi \sim \exp\{i(p^*t + a^*x + b^*z)\},$$

where a^*, b^* and p^* have been scaled in terms of the diffusion lengthscale $d^* = [\nu\kappa_T/\beta g(dS^*/dz^*)]^{\frac{1}{4}}$, where dS^*/dz^* is the imposed vertical salinity gradient. We cannot, of course, scale these quantities on the depth of the fluid, because that is now infinite. It is easily shown that

$$a^* = R_s^{-\frac{1}{4}}a, \quad b^* = R_s^{-\frac{1}{4}}b, \quad p^* = R_s^{-\frac{1}{2}}p, \tag{12}$$

where b replaces π in the preceding analysis. Equation (9) still holds with a, π, p written in terms of a^*, b^*, p^*, but (10) is different. The minimum value of λ occurs as $a^*, b^* \to 0$ with b^*/a^* bounded, i.e. the wavelength of the disturbance can become indefinitely large. In that case

$$\lambda_c = \frac{\sigma+\tau}{\sigma+1}, \quad p_c^{*2} = \sigma(1-\lambda_c)\left(1+\frac{b^{*2}}{a^{*2}}\right)^{-1}. \tag{13a, b}$$

For finite values of R_s, the critical value of λ is lower for an unbounded fluid than for a bounded fluid with stress-free boundaries. The results are, however, identical in the limit $R_s \to \infty$. Perhaps the most useful result for our present purposes is that the critical values of the horizontal and vertical wavenumbers are zero on the diffusion scale.

4. Linear stability with variable salinity gradient and $R_s \gg 1$

We now turn to a discussion of the linear stability of a layer of fluid in which the vertical salinity gradient G of the undisturbed state varies with depth z below the upper surface. We shall derive an asymptotic solution for $R_s \gg 1$, but first it is helpful to state some of the salient results of Zangrando and Bertram's calculations:

(i) the disturbance is essentially confined to the neighbourhood of a critical depth, denoted here by z_c;

(ii) the solution oscillates with $z - z_c$ and appears to decay exponentially as $|z - z_c|$ increases;

(iii) as R_s increases the vertical and horizontal scales of the motion decrease (this explains why it was difficult to obtain convergence at large values of R_s);

(iv) the critical value of λ appears to approach the value $(\sigma+\tau)(\sigma+1)^{-1}G_0$ as R_s increases, where G_0 is the value taken by G at $z = z_c$;

(v) G has a local minimum at $z = z_c$.

These results, together with (11 *b*) and (12), suggest that we seek a solution of (6) in the form

$$\psi = \exp\{i(pt+ax)\}\,\bar{\psi}(z), \tag{14}$$

where

$$p = R_s^{\frac{1}{2}}\bar{p}, \quad a = R_s^{\alpha}\bar{a}, \quad z = z_c + R_s^{-\beta}\bar{z}; \tag{15}$$

$\alpha, \beta > 0$; $\bar{p}, \bar{a}, \bar{z}$ are $O(1)$ as $R_s \rightarrow \infty$. The scaling adopted for p is the same as that for constant gradients as $R_s \rightarrow \infty$ whether or not the depth is finite. But α, β remain to be determined subject to the constraints. The salinity gradient G may be expanded in a Taylor series about z_c as follows:

$$G = G_0 + R_s^{-\beta}\bar{z}G_0' + \tfrac{1}{2}R_s^{-2\beta}\bar{z}^2G_0'' + O(R_s^{-3\beta}). \tag{16}$$

To leading order in R_s, G is constant, and the boundaries are at infinity on the scale of $\bar{z}$. The results given in §3 suggest that the critical value of λ is then $(\sigma+\tau)(\sigma+1)^{-1}G_0$, as indicated by Zangrando and Bertram's numerical results, and in agreement with Weinberger's (1964) criterion provided that the salinity gradient is evaluated at the critical depth. The results given in §3 also suggest that the horizontal and vertical scales of the motion are infinite on the diffusion scale and that their ratio is bounded. This means that

$$0 < \beta \leqslant \alpha < \tfrac{1}{4}. \tag{17}$$

Let us now substitute (14)–(16) into (6). We find that $\bar{\psi}$ satisfies

$$(i\bar{p}-\epsilon\bar{\nabla}^2)(i\bar{p}-\sigma\epsilon\bar{\nabla}^2)(i\bar{p}-\tau\epsilon\bar{\nabla}^2)\bar{\nabla}^2\bar{\psi} = \sigma\bar{a}^2\{i\bar{p}(G-\lambda)-\epsilon(G-\tau\lambda)\bar{\nabla}^2-2\epsilon\delta\mu G'\bar{D}-\epsilon\delta\mu^2G''\}\bar{\psi}, \tag{18}$$

where $\bar{D} \equiv d/d\bar{z}$, $\bar{\nabla}^2 \equiv \delta\bar{D}^2-\bar{a}^2$, and

$$\epsilon = R_s^{2\alpha-\frac{1}{2}}, \quad \mu = R_s^{-\beta}, \quad \delta = R_s^{2(\beta-\alpha)}.$$

Primes now denote differentiation of G with respect to z. In view of the restrictions on α, β made in (17) the quantities ϵ, δ, μ are all vanishingly small as $R_s \rightarrow \infty$. We wish to find solutions of (18) for which $\bar{p}$ is real and $\bar{\psi}$ tends exponentially to zero as $|\bar{z}| \rightarrow \infty$. We do so by expanding in powers of the small parameters ϵ, δ, μ. Rather than attempt an exhaustive and systematic search of the various balances that are possible between these small quantities, we put forward certain arguments that narrow down the options open to us.

If we seek to balance the leading algebraic term on the right of (18) – a term in $\bar{z}$ – with the leading derivative in $\bar{z}$ – the second – we obtain Airy's equation. It is well known that this equation has no solutions that have the desired property of tending exponentially to zero as $|\bar{z}| \rightarrow \infty$. We conclude that the coefficient of the term in $\bar{z}$ is zero, and this requires

$$G_0' = 0, \tag{19}$$

This result means that the critical depth z_c is defined to be the depth at which the vertical salinity gradient has a turning point. Since the salinity gradient has a stabilizing effect on the disturbance and since the temperature gradient is independent of the depth, physical arguments suggest that G has a minimum at $z = z_c$. This agrees with Bertram's computations.

The next most important algebraic term on the right of (18) is proportional to $\mu^2\bar{z}^2$. It is of the same order of magnitude as the leading $\bar{z}$-derivative if we choose $\delta = \mu^2$ or, equivalently, $\beta = \frac{1}{2}\alpha$. Then we have

$$\epsilon = R_s^{2\alpha-\frac{1}{2}}, \quad \mu = R_s^{-\frac{1}{2}\alpha}, \quad \delta = R_s^{-\alpha}. \tag{20}$$

We now expand in powers of the two remaining independent small parameters ϵ, μ by writing

$$(\bar{\psi}, \lambda, \bar{p}, \bar{a}^2) = \sum_m \sum_n \epsilon^m \mu^n (\bar{\psi}_{mn}, \lambda_{mn}, \bar{p}_{mn}, \bar{a}^2_{mn}). \tag{21}$$

When (16), (20), (21) are substituted into (18) and leading terms $(O(\epsilon^0\mu^0))$ are equated to zero, we find that

$$\bar{p}^2_{00} = \sigma(G_0 - \lambda_{00}). \tag{22}$$

When $G_0 = 1$ this result corresponds to the equivalent result (13*b*) for an infinite layer in which $b^*/a^* = 0$. Terms $O(\epsilon^1\mu^0)$ give

$$3i\bar{p}^2_{00}\bar{p}_{10}\bar{a}^2_{00} + i\bar{p}^3_{00}\bar{a}^2_{10} + (\sigma+\tau+1)\bar{p}^2_{00}\bar{a}^4_{00}$$
$$= i\sigma[(\bar{p}_{10}\bar{a}^2_{00} + \bar{p}_{00}\bar{a}^2_{10})(G_0 - \lambda_{00}) - \bar{p}_{00}\bar{a}^2_{00}\lambda_{10}] + \sigma\bar{a}^4_{00}(G_0 - \lambda_{00}). \tag{23}$$

The real part of this equation gives

$$(\sigma+\tau+1)\bar{p}^2_{00} = (G_0 - \tau\lambda_{00}). \tag{24}$$

Equations (22), (24) may be solved for $\lambda_{00}, \bar{p}_{00}$ to give

$$\lambda_{00} = \frac{\sigma+\tau}{\sigma+1}G_0, \quad \bar{p}^2_{00} = \sigma\frac{1-\tau}{1+\sigma}G_0. \tag{25a, b}$$

The result for λ_{00} agrees with that suggested by our earlier heuristic argument and also with Zangrando and Bertram's numerical results. It is equivalent to Weinberger's criterion (2), in which the salinity gradient is evaluated at the critical depth.

The imaginary part of (23) gives

$$\bar{a}^2_{00}\bar{p}_{10}(3\bar{p}^2_{00} - (G_0 - \lambda_{00})) + \bar{a}^2_{10}\bar{p}_{00}(\bar{p}^2_{00} - \sigma(G_0 - \lambda_{00})) = -\sigma\bar{p}_{00}\bar{a}^2_{00}\lambda_{10}.$$

When (22) is used to eliminate $G_0 - \lambda_{00}$ we find that

$$\bar{p}_{00}\bar{p}_{10} = -\tfrac{1}{2}\sigma\lambda_{10}. \tag{26}$$

This is again consistent with (13*b*) with $b^*/a^* = 0$.

There are no forcing terms $O(\mu)$, so the next most important terms are $O(\epsilon^2)$, $O(\mu^2)$. These terms are of the same order of magnitude if $\epsilon = \mu$ or, equivalently, $\alpha = \frac{1}{5}$. With this choice of scaling it may be shown that the critical value of $\bar{a}_{00}$ at which λ_{02} takes its minimum value is infinity. This means that the critical value of the horizontal wavenumber is infinite on the $R_S^{\frac{1}{5}}$ scale, but, as we have shown in §3, it is zero on the diffusion scale $R_S^{\frac{1}{4}}$. An intermediate scaling is clearly needed, and this is obtained by balancing the terms in μ^2 with those in ϵ^3. This requires $\alpha = \frac{3}{14}$, which means that the scale for the horizontal wavenumber is $R_S^{\frac{3}{14}}$, and this lies in the range $(R_S^{\frac{1}{5}}, R_S^{\frac{1}{4}})$ as required. (The alternative choice of balancing μ^3 with ϵ^2 gives $\alpha = \frac{2}{11}$, which is too small.)

We now have

$$p = R_S^{\frac{1}{2}}\bar{p}, \quad a = R_S^{\frac{3}{14}}\bar{a}, \quad z - z_c = R_S^{-\frac{3}{28}}\bar{z}, \tag{27}$$

and $\mu^2 = \epsilon^3 = R_S^{-\frac{3}{14}}$. With this choice of ϵ, μ we need to expand in powers of only one small parameter ϵ:

$$(\bar{\psi}, \lambda, \bar{p}, \bar{a}^2) = (\bar{\psi}_0, \lambda_0, \bar{p}_0, \bar{a}^2_0) + \epsilon(\bar{\psi}_1, \lambda_1, \bar{p}_1, \bar{a}^2_1) + \dots. \tag{28}$$

We now substitute (27), (28) into (18) and equate terms $O(\epsilon^m)$, $m = 0, 1, 2, \dots$, to zero. At leading order we recover (22), and at $O(\epsilon)$ we recover (23) in which the second suffix has been dropped. The results (25), (26) still hold for $\bar{p}_0, \lambda_0, \bar{p}_1$. At $O(\epsilon^2)$ we find that

$$2i(\sigma+1)\,\bar{a}_0^4\bar{p}_1+(\sigma+\tau+\sigma\tau)\,\bar{a}_0^6-2\bar{a}_0^2\bar{p}_0\bar{p}_2-\bar{a}_0^2\bar{p}_1^2=\sigma\lambda_2\bar{a}_0^2. \tag{29}$$

The imaginary part of this equation gives

$$\bar{p}_1=0, \tag{30}$$

and it follows from (26) that $\lambda_1 = 0$ also. The real part of (29) gives

$$2\bar{p}_0\bar{p}_2+\sigma\lambda_2\bar{p}_0=(\sigma+\tau+\sigma\tau)\,\bar{a}_0^4. \tag{31}$$

Terms $O(\epsilon^3)$ give a differential equation for $\bar{\psi}_0$:

$$[\bar{p}_0^3\bar{D}^2+\tfrac{1}{2}\sigma\bar{a}_0^2\bar{p}_0G_0''\bar{z}^2-iA]\,\bar{\psi}_0=0, \tag{32}$$

where $A = A_r + iA_i$, with

$$A_r=\sigma\tau\bar{a}_0^8-\sigma\tau\lambda_2\bar{a}_0^4-2(\sigma+\tau+1)\,\bar{a}_0^4\bar{p}_0\bar{p}_2,$$

$$A_i=-2\bar{a}_0^2\bar{p}_0^2\bar{p}_3-2\bar{a}_1^2\bar{p}_0^2\bar{p}_2+3(\sigma+\tau+\sigma\tau)\,\bar{a}_1^2\bar{p}_0\bar{a}_0^4-\sigma\lambda_2\bar{a}_1^2\bar{p}_0-\sigma\lambda_3\bar{a}_0^2\bar{p}_0.$$

The solution is

$$\bar{\psi}_0=\text{constant}\times\exp\{\tfrac{1}{2}ik_0\bar{z}^2\}, \tag{33}$$

where

$$k_0^2\bar{p}_0^2=\tfrac{1}{2}\sigma\bar{a}_0^2G_0'',\quad k_0\bar{p}_0^3=A. \tag{34a, b}$$

Since the layer is most unstable at $z = z_c$, the salinity gradient has a minimum there, and hence $G_0'' > 0$. It follows from (34a) that k_0 is real, in which case (34b) gives

$$k_0\bar{p}_0^3=A_r,\quad A_i=0. \tag{35a, b}$$

The solution for $\bar{\psi}_0$ is then bounded as $|\bar{z}| \to \infty$, but it does not decay exponentially as required. In fact this solution breaks down for large values of $|\bar{z}|$ because secular terms enter at higher order. This means that we need to postulate the existence of an outer region in which $|z-z_c| \gg \epsilon^{\frac{3}{2}}$, where, it is hoped, the solution does decay exponentially. Let us write

$$z-z_c=\epsilon^{\gamma}z, \tag{36}$$

where $0 < \gamma < \frac{3}{2}$ and $\epsilon = R_s^{-\frac{1}{14}}$ as before. In this wider, outer, region we suppose that ψ_0 varies on the scale of z^* and also on the shorter scale of $\bar{z}$. Let us write

$$\psi=\psi^*(z^*)\exp\{\tfrac{1}{2}ik_0z^{*2}\epsilon^{2\gamma-3}\}. \tag{37}$$

The solution given in (33) is incorporated into this expression as a rapid oscillation of wavelength $O(\epsilon^{3-2\gamma})$ on the z^* scale. Using this representation for ψ, we find that

$$\bar{\nabla}^2\psi\equiv\exp\{\tfrac{1}{2}ik_0z^{*2}\epsilon^{2\gamma-3}\}\,[-\bar{a}^2-\epsilon^{2\gamma}k_0^2z^{*2}+i\epsilon^3k_0L^*+\epsilon^{6-2\gamma}D^{*2}]\,\psi^*(z^*), \tag{38}$$

where $D^* \equiv d/dz^*$ and $L^* = 2z^*D^*+1$. The parameter γ that determines the scaling for this region is obtained by balancing the leading z^* derivative in (18) with the leading algebraic term on the right-hand side of (18). The term in $\epsilon^{2\gamma}z^{*2}$ has already been taken care of by the exponential term in (37), which leaves terms in $\epsilon^{2\gamma+1}z^{*2}$ and $\epsilon^{3\gamma}z^{*3}$. From (38) it can be seen that the leading z^* derivative is $O(\epsilon^3)$. A balance of all three of these terms is then obtained by taking $\gamma = 1$. We shall show below that this choice of γ gives a solution that decays exponentially as $|z^*| \to \infty$.

Having determined the new scaling for z we now proceed as before to set up an expansion in powers of ϵ. Fortunately we do not need to write down terms $O(\epsilon^0)$, $O(\epsilon^1)$ because these are the same as before, and terms $O(\epsilon^2)$ differ from those given in (29) only by the inclusion of two terms in z^{*2}, which cancel each other out. At $O(\epsilon^3)$, however, we find a new differential equation for ψ_0^*:

$$\{p_0^{-3}k_0L^*+A_1^*z^{*2}+iA_2^*z^{*3}-A\}\,\psi_0^*=0, \tag{39}$$

where A is given in (32), and

$$A_1^* = -\tfrac{1}{2}\sigma G_0'' \bar{a}_0^4 + (\sigma+\tau+1)\, k_0^2 \bar{p}_0^2 \bar{a}_0^2 - i\bar{a}_0^2 \bar{a}_1^2 \bar{p}_0^3 k_0^2,$$

$$A_2^* = -\tfrac{1}{6}\sigma \bar{a}_0^2 \bar{p}_0\, G_0'''.$$

The required solution is

$$\psi_0^* = \text{constant} \times \exp\{-\tfrac{1}{2}k_{11} z^{*2} + \tfrac{1}{3} i k_{12} z^{*3}\}, \tag{40}$$

where $$2\bar{p}_0^3 k_0 k_{11} = A_1^*, \quad 2\bar{p}_0^3 k_0 k_{12} = A_2^*, \quad \bar{p}_0^3 k_0 = A. \tag{41 a, b, c}$$

Equation (41c) is identical with (34b), while the real part of (41a) gives

$$\mathrm{Re}\{k_{11}\} = \frac{k_0^2(\tau+1+\sigma)\,\bar{p}_0^2 \bar{a}_0^2 - \tfrac{1}{2}\sigma \bar{a}_0^4 G_0''}{2k_0 \bar{p}_0^3}.$$

This expression may be simplified using (25), (34) to give

$$\mathrm{Re}\{k_{11}\} = \frac{\sigma(\tau+\sigma)\,\bar{a}_0^4 G_0''}{4\bar{p}_0^3 k_0}. \tag{42}$$

For a solution that decays exponentially as $|z^*| \to \infty$ we need $\mathrm{Re}\{k_{11}\} > 0$, and since $G_0'' > 0$ we need $\bar{p}_0 k_0 > 0$. This means that we need to take the positive square root in (34a), i.e.

$$k_0 \bar{p}_0 = (\tfrac{1}{2}\sigma \bar{a}_0^2 G_0'')^{\frac{1}{2}}. \tag{43}$$

We may now eliminate k_0 from (35a) using (43), and at the same time eliminate $\bar{p}_2$ using (31) and $\bar{p}_0^2$ using (25b). The result is an equation connecting only two unknown quantities $\lambda_2, \bar{a}_0$. We find that λ_2 satisfies

$$\lambda_2 = \lambda_{21}|\bar{a}_0|^{-3} + \lambda_{22}|\bar{a}_0|^4, \tag{44}$$

where $$\lambda_{21} = (1-\tau)\,(\tfrac{1}{2}\sigma G_0'')^{\frac{1}{2}}\,(1+\sigma)^{-2} G_0,$$

$$\lambda_{22} = (\sigma + \tau(\tau+1+\sigma))/\sigma.$$

The ratio λ, of thermal Rayleigh number R to saline Rayleigh number R_s is then given by

$$\lambda = \frac{\sigma+\tau}{\sigma+1} G_0 + \epsilon^2 \lambda_2 + O(\epsilon^3), \tag{45}$$

where $\epsilon^2 = R_s^{-\frac{1}{7}}$ and λ_2 is given by (44). We need to know the minimum value of λ for which solutions exist, and we must therefore find the minimum of the expression on the right of (44) as a function of $|a_0|$. A simple calculation yields

$$\lambda_{2\min} = \tfrac{7}{4}\lambda_{21}\left(\frac{4\lambda_{22}}{3\lambda_{21}}\right)^{\frac{3}{7}}, \tag{46}$$

and the minimum occurs when

$$|\bar{a}_0| = |\bar{a}_0|_{\text{crit}} = (3\lambda_{21}/4\lambda_{22})^{\frac{1}{7}}. \tag{47}$$

In terms of the original scaling the critical value of the horizontal wavenumber $\bar{a}$ is given to a first approximation by

$$|a|_{\text{crit}} = R_s^{\frac{3}{14}}[|\bar{a}_0|_{\text{crit}} + O(R_s^{-\frac{1}{14}})]. \tag{48}$$

The critical value p_{crit} of the frequency of the overstable oscillations is

$$p_{\text{crit}} = R_s^{\frac{1}{2}}\left\{\left[\sigma\frac{1-\tau}{1+\sigma}G_0\right]^{\frac{1}{2}} + R_s^{-\frac{1}{7}} p_{2\text{crit}} + \dots\right\}, \tag{49}$$

where $p_{2\text{crit}}$ is connected to $\lambda_{2\text{crit}}$ and $a_{0\text{crit}}$ through (31). The disturbance is confined to a region of vertical extent $O(R_s^{-\frac{1}{14}})$ centred at the critical depth z_c, where G has a local minimum. The form taken by ψ in this region may be described as a rapid oscillation subject to a relatively slow exponential decay.

The solution described above agrees in general terms with the five features of Zangrando and Bertram's computed results listed at the beginning of this section. A more detailed comparison of our results will be given in §5.

5. Discussion

The results obtained in §4 allow us to state a criterion for the stability of a layer of fluid stratified by heat and salt when the thermal and saline gradients are large and only the vertical temperature gradient is constant. If G_0 is the *minimum* value of the salinity gradient in the layer, then the layer is stable to small perturbations if

$$\frac{R}{R_s} = \lambda < \lambda_{\text{crit}} = \lambda_0 + R_s^{-\frac{1}{7}}\lambda_{2\text{min}} + \dots, \tag{50}$$

where $\lambda_0 = (\sigma+\tau)(\sigma+1)^{-1}G_0$ and $\lambda_{2\text{min}}$ is given by (46). For $R_s \to \infty$ this criterion reduces to a form similar to Weinberger's criterion (2), with the important difference that the relevant value of the salinity gradient is the minimum value and not a value averaged over the whole layer. Since $\lambda_{2\text{min}} > 0$ the layer is slightly more stable for finite values of R_s than for $R_s = \infty$.

An important feature of the solution is that the disturbance is confined to the neighbourhood of the critical depth at which the salinity gradient reaches its minimum value. The horizontal boundaries $z = 0, 1$ are at infinity on the vertical scale of the disturbance which suggests that the solution is independent of the depth d of the fluid layer. The choice of d as a lengthscale facilitates comparison of the results with observations and with Zangrando and Bertram's numerical results, but it would perhaps be more satisfactory from an analytic viewpoint to choose a more relevant lengthscale. A suitable candidate would be the radius of curvature of the salinity distribution at the critical depth. Equivalent results based on this scaling are given in the appendix. It should be noted that the values of R, R_s relevant to solar ponds are large whichever lengthscale is adopted.

The criterion (50) is particularly useful in determining the stability of solar ponds, whether man-made or natural. The most detailed observations of a man-made solar pond are given by Zangrando (1979). Several instances of instability were observed, and the temperature and salinity concentrations were recorded both before and after the instability occurred. We shall discuss one typical set of observations reproduced as figure 1. It was observed that the instability occurred at a depth of about 60 cm below the surface, and took the form of a step-like profile in the immediate neighbourhood of that depth. Zangrando approximated the temperature T^* over a sublayer extending from 50–75 cm below the surface by a quadratic,

$$T^* = 55{\cdot}0 + 11{\cdot}0(1{\cdot}3z - 0{\cdot}341z^2) \quad (0 \leqslant z \leqslant 1), \tag{51a}$$

and the salinity S^* by a cubic,

$$S^* = 12{\cdot}1 + 1{\cdot}6(1{\cdot}26z - 1{\cdot}82z^2 + 1{\cdot}56z^3) \quad (0 \leqslant z \leqslant 1). \tag{51b}$$

Using figure 1 it seems to us than an equally good fit to the observed profiles is

$$T^* = 56{\cdot}5 + 12{\cdot}0z, \quad S^* = 12{\cdot}0 + 1{\cdot}6(1{\cdot}21z - 1{\cdot}25z^2 + 1{\cdot}04z^3), \tag{52a, b}$$

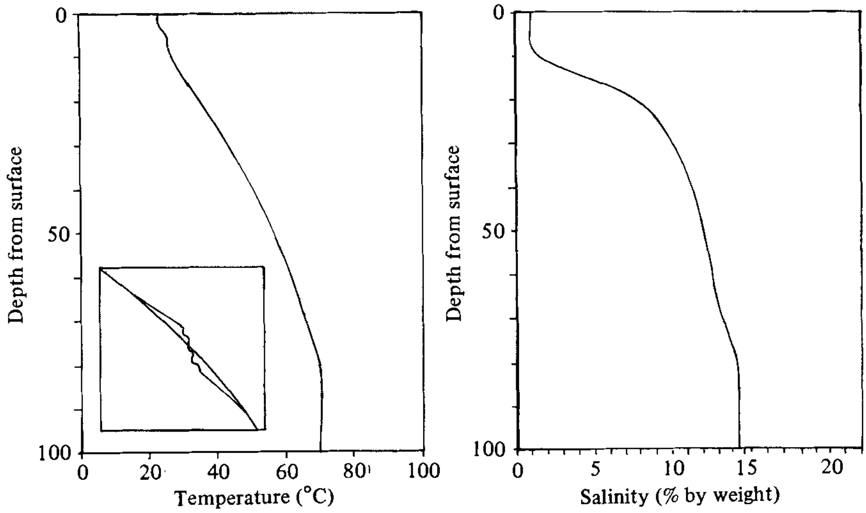

FIGURE 1. Temperature and salinity distributions on 22 May 1978, in the upper 100 cm of a solar pond. The insert represents the small convective layers established by 23 May 1978. (Taken from Zangrando 1979.)

in which T^* is approximated by a linear function of z, so that our theory may be applied. Using (52*a*) and Zangrando's salinity profile (51*b*) we find that $z_c = 59{\cdot}7$ cm and $\lambda_c = 0{\cdot}45$, whereas using (52*a*, *b*) we find that $z_c = 60{\cdot}0$ cm and $\lambda_c = 0{\cdot}58$. In both cases the prediction of the critical depth is excellent, but the critical value of λ is strongly dependent on the salinity profile. The accuracy of the salinity data is crucial to an estimate of λ_c, and, as Zangrando has pointed out to us, it is the most difficult quantity to estimate in operating ponds. The value of R_s for these observations is 10^{10}, and the first approximation to λ_c is 0·56, using the profiles (52). A more detailed and accurate analysis of the observational data is clearly needed before any firm conclusion can be drawn here, but our theory does seem to give promising results. The correction to the first approximation to λ_{crit} is not more than a few per cent, which suggests that Weinberger's criterion, suitably interpreted, provides an excellent operating criterion for the stability of solar ponds.

A survey of the non-convecting regions of five natural solar ponds has been given by Hauser (1976). In each system, $R_s \sim 10^{10}$–10^{15}, $\lambda < 0{\cdot}1$ and $(\sigma+\tau)(\sigma+1)^{-1} = 0{\cdot}9$. Hauser made the quite-unjustifiable assumption that both temperature and salinity gradients could be approximated by a constant, in which case $G_0 = 1, \lambda_0 \approx 0{\cdot}9, \lambda_2 = 0$. The condition for stability (50) then reduces to Weinberger's criterion and is seen to be easily satisfied. Again a more detailed examination of the data is required before a good comparison can be made with the theory, but, even allowing for a minimum salinity gradient of half the average value over the layer, the stability criterion is still easily satisfied.

Let us now return to the numerical computations currently being undertaken by Zangrando and Bertram. They assumed that the vertical temperature gradient was linear and that the salinity gradient was variable: in most runs it was a quadratic with a minimum at the centre of the layer. The general features of their solution have already been discussed in §4, and we have demonstrated that our theoretical solution possesses the same features.

Excellent agreement has been achieved for the critical values of λ, p for $R_s = 10^8$

	R_s	λ_c	a_c^2	P
*	10^8	0·2497	$4{\cdot}78\times10^2$	$3{\cdot}74\times10^3$
+	10^8	0·2387	$9{\cdot}09\times10^2$	$4{\cdot}59\times10^3$
*	10^{12}	0·2249	$4{\cdot}04\times10^4$	$4{\cdot}41\times10^5$
+	10^{12}	0·2244	$4{\cdot}71\times10^4$	$4{\cdot}63\times10^5$

TABLE 1. Comparison of the asymptotic results (marked +) with numerical results (marked *) by Zangrando and Bertram, for $Pr = 7$, $\tau = \frac{1}{80}$, $G_0 = 0{\cdot}25$, $G_0'' = -18$

and 10^{12} and $G_0 = 0{\cdot}25$ (see table 1). Zangrando and Bertram hope to report fully on their results in due course and a detailed comparison with our asymptotic results will then be undertaken.

Comparisons of linear asymptotic theory with linear numerical results are quite valid, but we must be very wary about comparing such results with observations of experiments in which nonlinear effects may be important. The danger is especially great in this problem because Huppert & Moore (1976) have shown that, at least for the classical Baines & Gill model, the exchange-of-stabilities solution is subcritically unstable. Furthermore, Proctor (1981) has demonstrated that for $\tau \ll 1$, $R_S \tau^{\frac{1}{2}} \lesssim 1$ finite-amplitude monotonic solutions exist for thermal Rayleigh numbers below that predicted by linear theory for the onset of the oscillatory mode. This raises the possibility that solar ponds may become unstable to disturbances of small but finite amplitude at Rayleigh numbers well below those considered in this paper. However, Proctor's work is concerned with a layer with constant gradients in which R_s is taken to be $O(1)$, and his detailed results may not prove to be a true guide to the stability characteristics of a layer with variable gradients and $R_s \gg 1$.

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Appendix

In this appendix we show that the solution given earlier is independent of the depth d of the fluid layer, and derive an alternative solution based on ρ, the radius of curvature of the salinity concentration at the critical depth.

The thermal and saline Rayleigh numbers based on d are

$$R = \frac{\alpha g\,\Delta T/d}{\nu k_T} d^4, \quad R_s = \frac{\beta g\,\Delta S/d}{\nu k_T} d^4,$$

in which $\Delta T/d$, $\Delta S/d$ be regarded as typical temperature and salt gradients. Let us define equivalent Rayleigh numbers $R(\rho)$, $R_s(\rho)$ based on ρ. Then

$$R(\rho) = R\left(\frac{\rho}{d}\right)^4, \quad R_s(\rho) = R_s\left(\frac{\rho}{d}\right)^4.$$

If all lengths are now scaled on ρ instead of d, the results given earlier apply, but with minor modifications. First, the wavenumber a_0 should be replaced by $a_0^*(d/\rho)$, where a_0^* is the horizontal wavenumber in the new scalings, and λ_{21} should be replaced by $\lambda_{21}^*(d/\rho)$, $\lambda_{21}^* = (1-\tau)(\frac{1}{2}\sigma)^{\frac{1}{2}}(1+\sigma)^{-2}G_0$. Then

$$\begin{aligned}\lambda-\lambda_0 &= R_s^{-\frac{1}{7}}(\lambda_{21}|\bar{a}_0|^{-3}+\lambda_{22}|\bar{a}_0|^4)\\ &= R_s^{\frac{1}{2}}\lambda_{21}|a_0|^{-3}+R_s^{-1}\lambda_{22}|a_0|^4\\ &= R_s(\rho)^{\frac{1}{2}}\left(\frac{d}{\rho}\right)^2\lambda_{21}\frac{d}{\rho}|a_0|^3\left(\frac{\rho}{d}\right)^3+R_s^{-1}(\rho)\left(\frac{\rho}{d}\right)^4\lambda_{22}|a_0|^4\left|\frac{d}{\rho}\right|^4\\ &= R_s(\rho)^{\frac{1}{2}}\lambda_{21}|a_0|^3+R_s^{-1}(\rho)\,\lambda_{22}|a_0|^4.\end{aligned}$$

This is the same expression as before, but is now independent of d. The horizontal scale of the motion is

$$\begin{aligned}d|a_0|_{\mathrm{crit}}^{-1} &= dR_s^{-\frac{3}{14}}|\bar{a}_0|_{\mathrm{crit}}^{-1} = dR_s^{-\frac{3}{14}}\left(\frac{3\lambda_{21}}{4\lambda_{22}}\right)^{-\frac{1}{7}}\\ &= DR_s(\rho)^{-\frac{3}{14}}\left(\frac{\rho}{d}\right)^{\frac{6}{7}}\left(\frac{3\lambda_{21}^*}{4\lambda_{22}}\right)^{-\frac{1}{7}}\left(\frac{d}{\rho}\right)^{-\frac{1}{7}}\\ &= \rho R_s(\rho)^{-\frac{3}{14}}\left(\frac{3\lambda_{21}^*}{4\lambda_{22}}\right)^{-\frac{1}{7}},\end{aligned}$$

and again we arrive at the same expression as before, but with d replaced by ρ. Similar results apply to the vertical scale of the motion.

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